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LETTER TO THE EDITOR

On exact solitary wave solutions of the nonlinear Schrödinger equation with a source

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Abstract
We use a fractional transformation to connect the travelling wave solutions of the nonlinear Schrödinger equation (NLSE), phase locked with a source, to the elliptic equations satisfying, $f'' \pm af \pm \lambda f^3 = 0$. The solutions are *necessarily* of rational form, containing both trigonometric and hyperbolic types as special cases. Bright and dark solitons, as well as singular solitons, are obtained in a suitable range of parameter values.

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(Some figures in this article are in colour only in the electronic version)

Much attention has been paid to the study of the externally driven NLSE, after the seminal work of Kaup and Newell [1]. This equation features prominently in the problem of optical pulse propagation in asymmetric, twin-core optical fibres (TCF) [2–4], currently an area of active research. Of the several applications of an externally driven NLSE, perhaps the most important ones are to long Josephson junctions [5], charge density waves [6], plasmas driven by rf fields [7] and chaotic phenomena [8]. The phenomenon of auto-resonance [9, 10], indicating a continuous phase locking between the solutions of NLSE and the driving field, has been found to be a key characteristic of this system. In the presence of damping, this dynamical system exhibits rich structure including bifurcation. This is evident from analyses around a constant background, as well as numerical investigations [11–13]. Although the NLSE is a well-studied integrable system [14], no exact solutions have so far been found for the NLSE with a source, to the best of the authors' knowledge. All the above inferences have been drawn through perturbations around solitons and numerical techniques.

In this letter, we map *exactly* the travelling wave solutions of the NLSE phase locked with a source to the elliptic equations, through a fractional transformation (FT). It was found that the solutions are *necessarily* of rational type, with both the numerator and denominator containing terms quadratic in elliptic functions, in addition to having constant terms. It is well known that the solitary wave solutions of the NLSE [15, 16] are cnoidal waves, which contain localized soliton solutions in the limit, when the modulus parameter equals 1 [17]. Hence, the solutions found here, for the NLSE with a source, are nonperturbative in nature. We find both bright and dark solitons as well as singular ones. Solitons and solitary pulses show distinct behaviour. In the case when the source and the solutions are not phase matched, perturbation around these solutions may provide a better starting point.

For nonlinear equations, a number of transformations are well known in the literature, which map the solutions of a given equation to the other [18, 19]. The familiar example is the Miura transformation [20], which maps the solutions of the modified KdV to those of the KdV equation. To find static and propagating solutions, appropriate transformations have also been cleverly employed, to connect the nonlinear equations to the ones satisfied by the elliptic equations: $f'' \pm af \pm \lambda f^3 = 0$. Here and henceforth, prime denotes derivative with respect to the argument of the function. Solitons and solitary wave solutions of KdV, NLSE and sine-Gordon, etc can be easily obtained in terms of the elliptic functions in this manner.

The goal of this letter is to find the solutions of the NLSE, phase locked with a source, satisfying

$$\mathrm{i}\frac{\partial\psi}{\partial t} + \frac{\partial^2\psi}{\partial x^2} + g|\psi|^2\psi + \mu\psi = \kappa\,\mathrm{e}^{\mathrm{i}[\chi(\xi)-\omega t]}, \tag{1}$$

where g, μ and k are real and $\xi = \alpha(x - vt)$. The travelling wave solution is taken to be $\psi(x, t) = \mathrm{e}^{\mathrm{i}[\chi(\xi)-\omega t]}\rho(\xi)$. Separating the real and imaginary parts of equation (1), and integrating the imaginary part, one gets

$$\chi' = \frac{v}{2\alpha} - \frac{c}{2\rho^2}, \tag{2}$$

where c is the integration constant. In order that, the external phase is independent of ψ, we put $c = 0$ to obtain

$$\alpha^2\rho'' + g\rho^3 + \epsilon\rho - \kappa = 0, \tag{3}$$

where $\epsilon = \omega + \frac{v^2}{4} + \mu$.

We find, after a straightforward but lengthy algebra, that the following FT,

$$\rho(\xi) = \frac{A + Bf^\delta(\xi)}{1 + Df^\delta(\xi)}, \tag{4}$$

for $AD - B \neq 0$, maps the solutions of equation (3) to the elliptic equations: $f'' = af - \lambda f^3$, with a conserved quantity $E_0 = f'^2/2 + (1/4)\lambda f^4 - af^2/2$, provided $\delta = 2$. The elliptic equation has non-singular oscillatory solutions of the type $\mathrm{cn}(\xi, m)$, $\mathrm{sd}(\xi, m)$, $\mathrm{dn}(\xi, m)$ and $\mathrm{nd}(\xi, m)$, where m is the modulus parameter. The above also has localized soliton solutions for $m = 1$. It should be noted that for the attractive case ($g < 0$), the bounded solutions are $\mathrm{sn}(\xi, m)$ and $\mathrm{cd}(\xi, m)$. For $AD = B$, one only finds a constant amplitude solution.

Before elaborating on specific cases, a number of interesting features emerging from the above mapping is worth mentioning. First of all, the nontrivial solutions are necessarily of rational type, i.e., $D \neq 0$, in the presence of the source. Analysis of the consistency conditions emerging from the substitution of equation (4) in equation (3) imply that when $D = 0$, B and α are also necessarily zero. This indicates that A is a constant, since it satisfies $2E_0\alpha^2 + gA^3 + A\epsilon - \kappa = 0$. The same consistency condition also shows that $E_0 = 0$ and $E_0 \neq 0$ cases exhibit characteristically different behaviour. For example, for $E_0 = 0$ case, A decouples from the rest of the solution parameters and is a solution of the above cubic equation. The same is not true for the $E_0 \neq 0$ case. We further find that the restricted solutions corresponding to $A = 0$ and $B = 0$ cases, for which the equation parameters ϵ, κ and g need to satisfy a constraint relationship, also show different behaviour. The analysis below illustrates these features.

For explicitness, we consider equation (3), with all the parameters and illustrate below various type of solutions, taking $f = \mathrm{cn}(\xi, \mathrm{m})$, as an example. Other cases can be similarly worked out. The consistency conditions are given by

$$A\epsilon - 2\alpha^2(AD - B)(1 - m) + gA^3 - \kappa = 0, \tag{5}$$

$$2\epsilon AD + \epsilon B + 6\alpha^2(AD - B)D(1 - m) - 4\alpha^2(AD - B)(2m - 1) + 3gA^2B - 3\kappa D = 0, \tag{6}$$

$$A\epsilon D^2 + 2\epsilon BD + 4\alpha^2(AD - B)D(2m - 1) + 6\alpha^2(AD - B)m + 3gAB^2 - 3\kappa D^2 = 0, \tag{7}$$

$$\epsilon BD^2 - 2\alpha^2(AD - B)Dm + gB^3 - \kappa D^3 = 0. \tag{8}$$

The above equations clearly indicate that the solutions, for $m = 1$, $m = 0$ and other values of m, have distinct properties. For example, when $m = 1$, A is obtained as the solution of the cubic equation (equation (5)), containing the source strength k. Similarly, for $m = 0$, either B or D appears as the solution of equation (8). As noted earlier, when $D = 0$, B also equals zero, indicating only a constant solution ($A \neq 0$). One needs to be careful in choosing the real solutions of the above cubic equations, for a suitable range of the parameter values. Although a wide class of solutions is allowed, for brevity, we only outline a few of the interesting solutions and their properties. We start with the solutions for which the equation parameters are related in a specific manner and then give the general localized solution.

Case I (trigonometric solution). In the limit $m = 0$, unlike the unperturbed NLSE, in this case one finds rational solutions of the trigonometric type. Apart from the general solutions, interestingly, for these, one can obtain special cases where $A = 0$ and $B \neq 0$ is allowed. However, the converse is forbidden. The following is an example of this type of non-singular periodic solution, for the repulsive case:

$$\rho(\xi) = \left(-\frac{2\kappa}{\epsilon}\right) \frac{\cos^2(\xi)}{1 - \frac{2}{3}\cos^2(\xi)}. \tag{9}$$

Here, ϵ has to be negative since $\alpha^2 = -\epsilon/4$; it is given by $\epsilon = (-27g\kappa^2/2)^{1/3}$.

We note that the value of κ does not have to be restricted, as has been the case for studying the auto-resonance [9, 4] ($\kappa < 0.6$) in this type of model.

Case II (hyperbolic solution). Unlike the above periodic case, here one finds that the solutions with $B = 0$ and $A \neq 0$ are allowed, the converse not being true. Hence, these localized solitons behave differently from the soliton trains. In this case the solution is necessarily singular. For $B = 0$ and $m = 1$, we found that $\alpha^2 = \epsilon/4$ and $\epsilon = (-27g\kappa^2/2)^{1/3}$. This yields, the singular hyperbolic solution

$$\rho(\xi) = \left(\frac{3\kappa}{\epsilon}\right) \frac{1}{1 - \frac{3}{2}\,\mathrm{sech}^2(\xi)}. \tag{10}$$

This solution corresponds to the attractive case, i.e., $g < 0$.

The singularity of this pulse profile may correspond to the beam power exceeding the material breakdown due to self-focusing [21–24]. Interestingly, non-singular hyperbolic solutions of the above type are not present.

Case III (pure cnoidal solutions). For $0 < m < 1$, we list below a few special cases. For $A = 0$, $D = 1$ and $m = 5/8$; we have $\alpha^2 = (2/7)\epsilon$ and $\epsilon = 7(-g\kappa^2/18)^{1/3}$; this corresponds to the attractive case. Explicitly, the solution is given by

$$\rho(\xi) = \left(\frac{14\kappa}{3\epsilon}\right) \frac{\mathrm{cn}^2(\xi, m)}{1 + \mathrm{cn}^2(\xi, m)}. \tag{11}$$

For $A = 0$ and $m = 1/2$; it is found that $\alpha^2 = \epsilon/2\sqrt{3}$ and $\epsilon = (-27g\kappa^2)^{\frac{1}{3}}$, for which

$$\rho(\xi) = \left(\frac{2\sqrt{3}\kappa}{\epsilon}\right) \frac{\mathrm{cn}^2(\xi, m)}{1 + \frac{1}{\sqrt{3}}\mathrm{cn}^2(\xi, m)}. \tag{12}$$

Case IV (general localized solutions). For $m = 1$ we list below the general localized solutions. We note that for this case the equation for A does not involve B and D. We first solve this cubic equation (equation (5)), which is already in the Vieta form by using Cardano's formula. Thus for $A^3 + (\epsilon/g)A = \kappa/g$, the discriminant D is identified as $D = Q^3 + R^2$, with $Q = p/3$ and $R = q/2$, where $p = \epsilon/g$ and $q = \kappa/g$. As is known, different types of solutions may arise, depending on the parameter values. For example when $D < 0$, i.e., $\epsilon^3 < -27g\kappa^2/4$, there are three unequal real roots. By defining $\theta = \cos^{-1}(R/\sqrt{-Q^3})$, the real roots are $A_1 = 2\sqrt{-Q}\cos(\theta/3)$, $A_2 = 2\sqrt{-Q}\cos((\theta + 2\pi)/3)$ and $A_3 = 2\sqrt{-Q}\cos((\theta + 4\pi)/3)$.

Thus, A is determined in terms of ϵ, κ and g. From equation (6), we determine the value of D in terms of B as

$$D = \Gamma B,$$

where

$$\Gamma = \frac{\epsilon + 4\alpha^2 + 3gA^2}{4\alpha^2 A + 3\kappa - 2\epsilon A}.$$

By substituting this value in equation (7), the value of B is determined as

$$B = \frac{6\alpha^2(1 - A\Gamma)}{3gA + A\epsilon\Gamma^2 + 2\epsilon\Gamma + 4\alpha^2\Gamma(A\Gamma - 1) - 3\kappa\Gamma^2}.$$

From equation (8), we obtain a cubic equation in $\beta \equiv \alpha^2$:

$$p_1\beta^3 + q_1\beta^2 + r_1\beta + c = 0, \tag{13}$$

where

$$p_1 = 64(A^3 g + A\epsilon - \kappa), \qquad q_1 = (48A^5g^2 + 64A^3g\epsilon + 16A\epsilon^2 - 48A^2g\kappa - 16\epsilon\kappa),$$
$$r_1 = (12A^7g^3 + 36A^5g^2\epsilon + 20A^3g\epsilon^2 - 4A\epsilon^3 - 60A^4g^2\kappa - 72A^2g\epsilon\kappa + 4\epsilon^2\kappa + 48Ag\kappa^2)$$

and

$$c = (3A^7g^3\epsilon - 3A^5g^2\epsilon^2 - 7A^3g\epsilon^3 - A\epsilon^4 - 18A^6g^3\kappa - 15A^4g^2\epsilon\kappa + 12A^2g\epsilon^2\kappa + \epsilon^3\kappa + 9A^3g^2\kappa^2 - 15Ag\epsilon\kappa^2 + 9g\kappa^3).$$

Very interestingly, the coefficient p_1 in equation (13) is the consistency condition equation (5) for $m = 1$ and hence is identically zero. Therefore, the width parameter β is the solution of a quadratic equation. This completes the proof of our assertion about the existence of general localized solutions of the form

$$\rho = \frac{A + B\,\mathrm{sech}^2(\xi)}{1 + D\,\mathrm{sech}^2(\xi)}.$$

We have found various type of localized solutions, both dark and bright, depending on the values and signs of parameters A, B and D.

Since the localized solitons are usually robust, we have performed numerical simulations to check the stability of the solutions pertaining to case I, i.e., the trigonometric solution. It is worth pointing out that the numerical techniques based on the fast Fourier transform (FFT) are expensive as they require the FFT of the external source. Hence, we have used the Crank–Nicholson finite difference method [25] to solve the NLSE with a source (equation (1)),

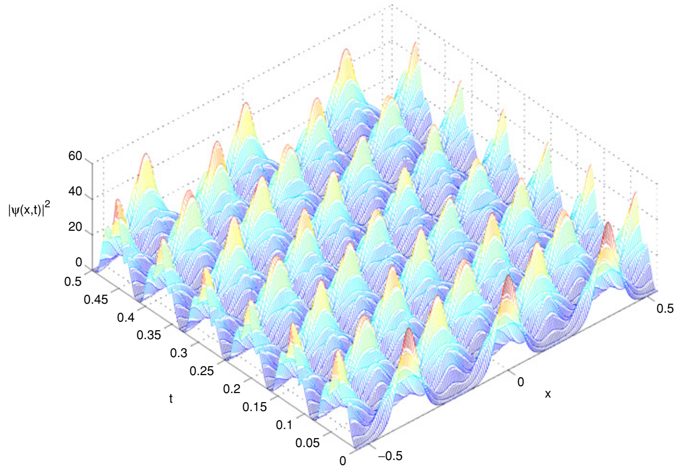

Figure 1. Nonlinear evolution of the unperturbed trigonometric solution for various times.

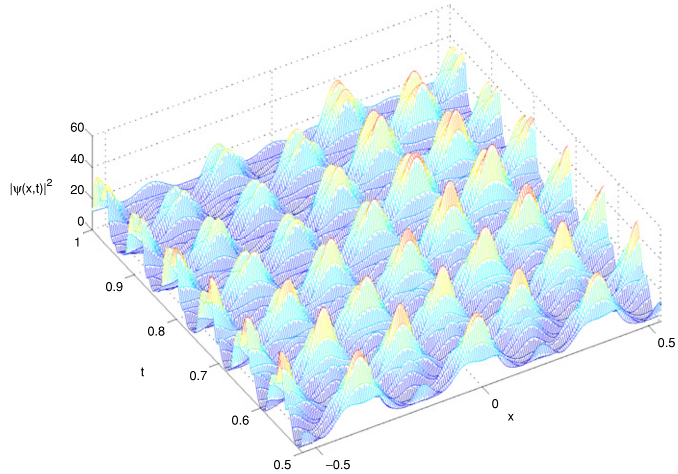

Figure 2. Nonlinear evolution of the perturbed trigonometric solution for various times.

which is quite handy, and unconditionally stable. To confirm this, we numerically study the nonlinear evolution of the exact solution, under small perturbation by directly simulating equation (1) with initial condition $\psi(x, t = 0) = \psi(x)[1+\epsilon'] \exp(\mathrm{i}\alpha x)$. This has been knitted on a lattice, with grid size $\mathrm{d}x = 0.005$ and $\mathrm{d}t = 5.0 \times 10^{-6}$. The nonlinear evolution of the same is depicted in figure 1 for the exact one and in figure 2 with a perturbation $\epsilon' = 0.2$, indicates that it almost remains stable as it propagates, although the peak of the intensity oscillates.

In conclusion, we have used a fractional transformation to connect the solutions of the phase-locked NLSE with the elliptic functions, in an exact manner. The solutions are necessarily of rational type that contain solitons, solitary waves, as well as singular ones. Our procedure is applicable both for the attractive and repulsive cases. Because of their exact nature, these will provide a better starting point for the treatment of general externally driven

NLSE. Considering the utility of this equation in fibre optics and other branches of physics, these solutions may find practical applications.

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